~~A gentle introduction to Arithmetic Complexity~~

A quick and gentle general introduction to Arithmetic Complexity
And a focus on the non-commutative setting

Guillaume Lagarde, LaBRI 18/04/24

Arithmetic Complexity?

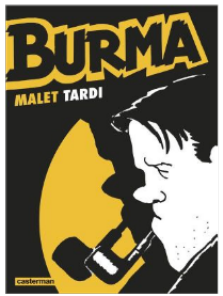
$$f(x_1, x_2, x_3, x_4) = 2 \cdot x_1^3 + 8 \cdot x_1^2 x_2 + 3 \cdot x_1^2 x_3 + 4 x_1^2 x_4$$
$$+ 10 \cdot x_1 x_2^2 + 9 \cdot x_1 x_2 x_3 + 10 \cdot x_1 x_2 x_4 + x_1 x_3^2$$
$$+ 4 x_1 x_3 x_4 + 2 x_1 x_4^2 + 4 x_2^3 + 6 x_2^2 x_3$$
$$+ 2 \cdot x_2 x_3^2 + 5 \cdot x_2 x_3 x_4 + 2 x_2 x_4^2$$
$$+ x_3^2 x_4 + x_3 x_4^2$$

Arithmetic Complexity?

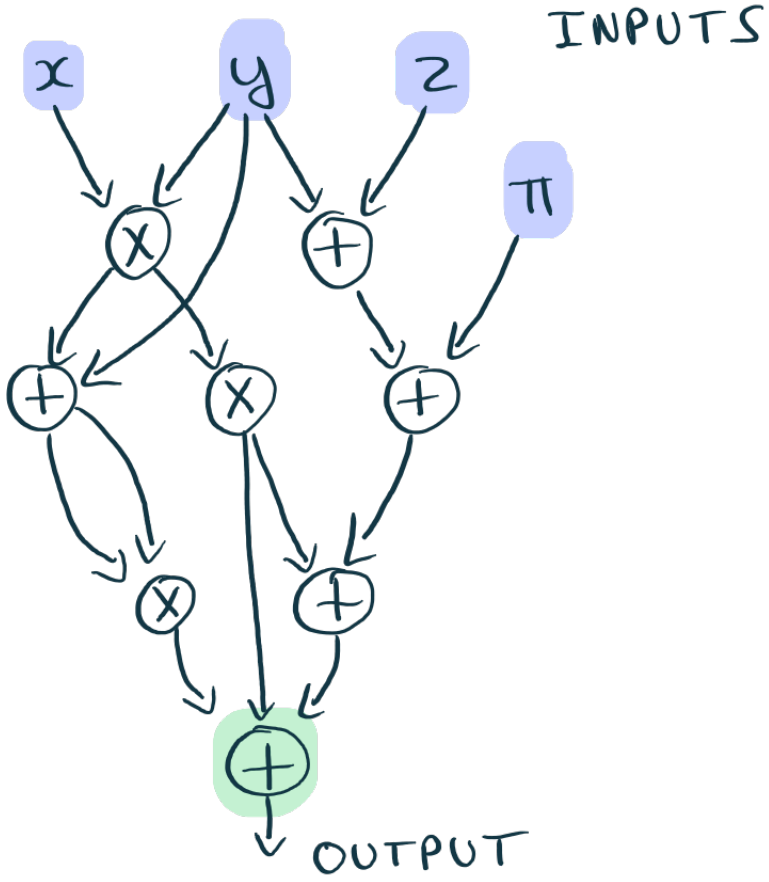
$$f(x_1,x_2,x_3,x_4) = 2\cdot x_1^3 + 8\cdot x_1^2 x_2 + 3\cdot x_1^2 x_3 + 4\,x_1^2 x_4 + 10\cdot x_1 x_2^2 + 9\cdot x_1 x_2 x_3 + 10\cdot x_1 x_2 x_4 + x_1 x_3^2 + 4\,x_1 x_3 x_4 + 2\,x_1 x_4^2 + 4\,x_2^3 + 6\,x_2^2 x_3 + 2\cdot x_2 x_3^2 + 5\cdot x_2 x_3 x_4 + 2\,x_2 x_4^2 + x_3^2 x_4 + x_3 x_4^2$$

$$= (2x_1 + 2x_2 + x_3)\cdot(x_1 + 2x_2 + x_4)\cdot(x_1 + x_2 + x_3 + x_4)$$

surely a more compact representation



Arithmetic Circuits



A gate = a formal polynomial

SIZE = number of gates/edges

DEPTH = longest path

- Natural model to compute polynomials
- Compact representations
- Algebraic variant of Boolean complexity, with more structure

Boolean World

- boolean functions
- Turing Machines, boolean circuits, branching programs, ...
- Turing reductions
- P, NP, BPP, NC, #P, ...
- $P = NP$? i.e., $SAT \in P$?

Algebraic World

- polynomials
- arithmetic circuits, algebraic branching programs, ...
- projections
- VP, VNP, VF, VNC, ...
- $VP = VNP$? $\approx$ DET vs. PERM?

Circuits

Find efficient algorithms to manipulate the circuits

PIT

Input: a circuit C

Output: is $C \equiv 0$?

Polynomials

For a polynomial, is there a circuit such that...

Lower Bounds

Find (explicit) polynomials that require large circuits

Deterministic polynomial-time algorithms in the size of C.

WANTED

Superpolynomial lower bounds in the number of variables and the degree

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Kabanets-Impagliazzo [2003]

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Polynomial randomized algorithm

No deterministic subexponential time yet

REALITY

Baur and Strassen [1983]

$\sum_{i=1}^{n} x_i^d$ requires circuits of size $\Omega(n \log d)$

STILL HOPE

* monotone formula — $2^{\Omega(n)}$ — Nisan '91
* homogeneous depth-3 circuit — $2^{\Omega(n)}$ — Nisan-Wigderson '97
* multilinear formula — $2^{\Omega(n \log n)}$ — Raz '09
* constant-depth circuits — $n^{(\log n)^{\exp(-\Delta)}}$ (Δ = depth) — Limaye-Srinivasan-Tavenas '21

\+ many others (non-commutative, set-multilinear, ...)

Blackboard

* Baur & Strassen.

  → $\Omega(n \cdot \log d)$ lower bound

* Elementary Symmetric polynomials

$$X = \{x_1, \ldots, x_n\}$$

$$S_n^d = \sum_{\substack{T \subseteq X \\ |T|=d}} \left(\prod_{x \in T} x \right)$$

$$S_3^2 = x_1 \cdot x_2 + x_1 \cdot x_3 + x_2 \cdot x_3$$

Non-commutative Setting

- $F\langle x_1, x_2, \ldots, x_n \rangle$: ring of non-commutative polynomials

$$x_i \cdot x_j \neq x_j \cdot x_i \longrightarrow$$ Need to order the children of $\otimes$-gates

Motivations

- evaluating polynomials over non-commutative domains
- non-commutative algorithms $\Longrightarrow$ commutative algorithms
- commutative algorithms $\overset{\text{Lifting}}{\Longrightarrow}$ "non-commutative algorithms not too expensive"

No better lower bounds for general nc-circuits

$\Omega(n \cdot \log d)$ is the best so far

No better lower bounds for general nc-circuits

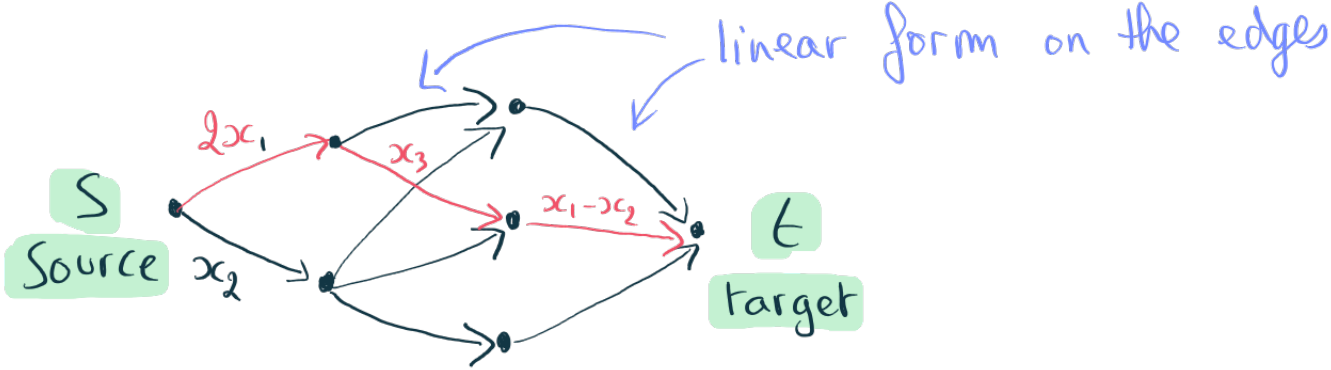
But for **Algebraic Branching Programs**

→ Exact characterization of the complexity

→ Exponential lower bounds for PERMANENT

Nisan '91

ABP (Branching Programs)



$$w(\nearrow \cdot \searrow \rightarrow \cdot) = (2x_1) \times (x_3) \times (x_1 - x_2)$$

$$f_{ABP} = \sum_{\substack{\text{path} \\ s \rightsquigarrow t}} w(\text{path})$$

→ captures Matrix Multiplication ! (and Determinant)

High level idea to prove lower bound

Measure $\mu : F\langle X \rangle \longrightarrow \mathbb{R}$

1. $\forall\ C \in \mathcal{C},\ \mu(C) = |C|^{O(1)}$
2. $\exists$ (explicit) polynomial p of high measure

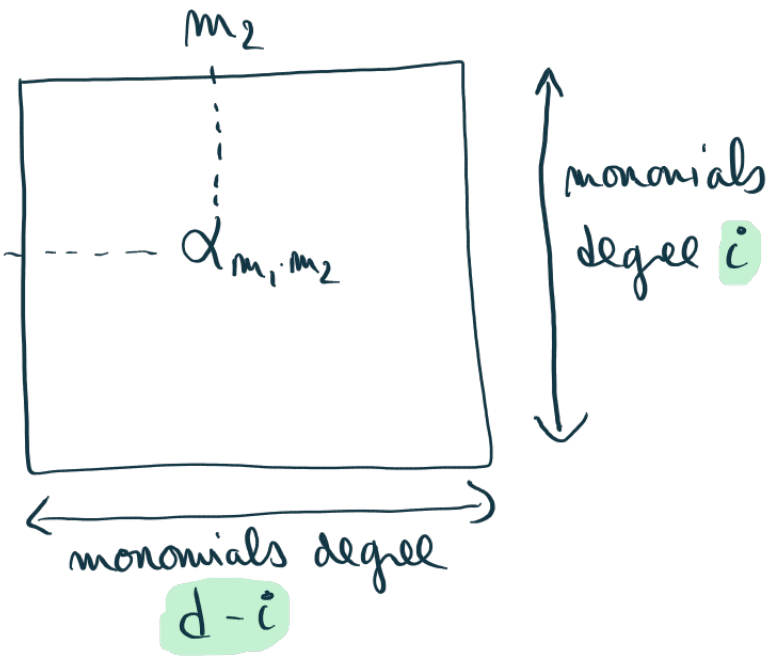
$$\Longrightarrow \quad |C|^{O(1)} \geq \mu(p)$$

$\mu(p)$: high

In Nisan, μ = rank of the partial derivative matrix

$$f = \sum_{m \in \{x_1, \ldots, x_n\}^d} \alpha_m \cdot m$$

$M^i[f]$ =



Theorem

The smallest ABP computing f homogeneous, degree d, has size

$$\sum_{i=0}^{d} rk(M^i[f])$$

Blackboard

- Palindrome

$$\sum_{|m|=\frac{n}{2}} m \cdot \overline{m}$$

mirror of m

- Proof Nisan's theorem

Extensions of Nisan's result

ABP ≡ circuit where monomials are computed with the shape 

Extensions of Nisan's result

ABP ≡ circuit where monomials are computed with the shape 

UPT any shape  but a unique one

Skew $\exp(d)$ shapes, rotations of 

rot-UPT $\exp(d)$ shapes, rotations of any tree 

k-pT any k shapes, lower bound for $k \leq 2^{d^{1/4}}$

Superpolynomial lower bounds ✓

general circuit ≡ $2^{O(d)}$ − pT OPEN!

Nisan's theorem 1991

Theorem

The smallest ABP computing f homogeneous has size

$$\sum_{i=0}^{d} rk(M^i[f])$$

looks similar

Fliess' theorem 1974

Theorem

word series $f : \Sigma^* \longrightarrow \mathbb{K}$

the smallest weighted automaton that recognises f has size

$$rk(Hankel[f])$$

Lousco - Bozapaladis 1983

extension of Fliess' theorem to trees

inspire

?

HANKEL MATRIX ON TREES

→ Unify almost all previous lower bounds !

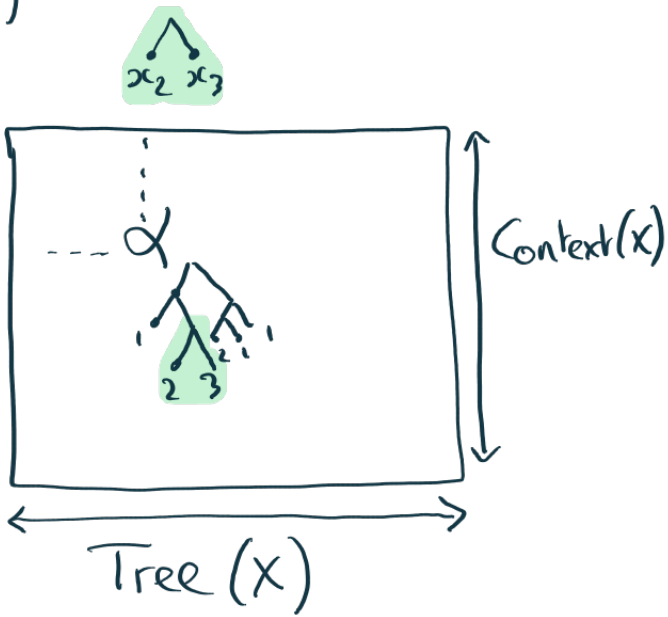
Circuits seen as non-associative (commutative or not)

$$f : \mathrm{Tree}(X) \to \mathbb{K}$$

"

$$f = \sum_{t \in \mathrm{Tree}(X)} \alpha_t \cdot t$$

$H[f] =$



THEOREM

The minimum circuit that computes f non-associative, homogeneous, has size

$$\mathrm{rk}(H[f])$$

→ gives a way to get superpolynomial lower bounds for circuits with $k = 2^{d^{1-\varepsilon}}$

↑ associative non-commutative & commutative

BEFORE, $k = 2^{d^{1/3}}$ → non-commutative & commutative

$k = \sqrt{d}$ → commutative

GENERAL CIRCUITS $k = 2^{o(d)}$

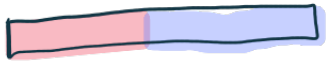
WANTED

For f associative, we want a lower bound on

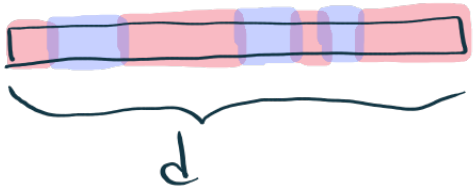
$$\min_{\tilde{f}\ \text{non-ass}} \operatorname{rk}\left(H[\tilde{f}]\right)$$

$\tilde{f}$ projects to f

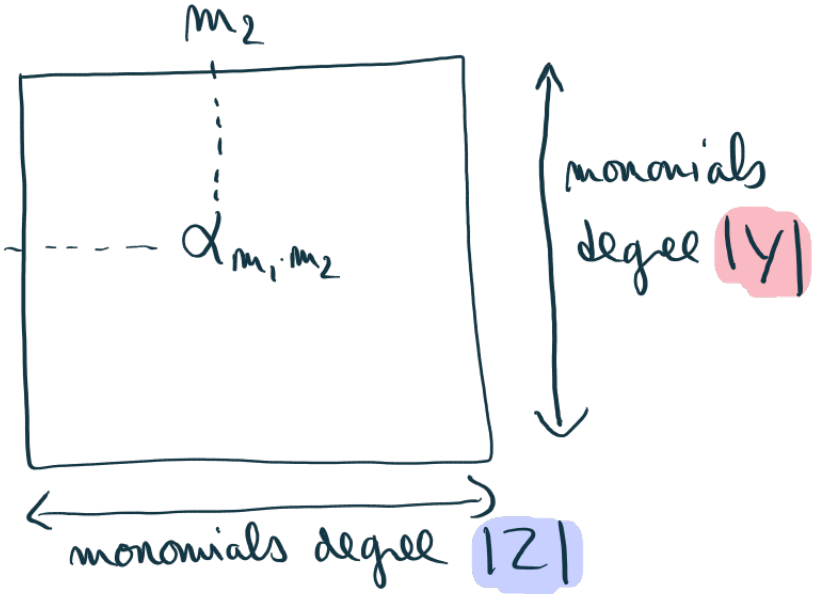
PARTITION METHOD

Nisan : $\pi =$ 

$\pi = (Y, Z)$ partition of $[d]$



$M^{\pi}[f] = m_1$



GOAL . Say we want a lower bound against a class $\mathcal{C}$.

- $\forall C \in \mathcal{C}$, $\exists \Pi$ s.t $rk(M^{\Pi}[C]) \leq poly(|C|)$
- $\exists f$ s.t for the same Π, $rk[M^{\Pi}[f]]$ large

Limaye, Malod, Srinivasan 2016

$\exists f$ homogeneous, full-rank w.r.t any partition Π computable by a polysize circuit.

f homogeneous computed by a circuit with R possible tree shapes

1. **Decomposition Lemma** (Thanks to the Hankel matrix!)

$$f \simeq \sum_{\text{small}} f_i$$

f_i not robust under random partition Π

2. f_i not robust $\quad \Pr_{\Pi \text{ random}}\left(\operatorname{rk}(M^{\Pi}[f_i]) \text{ large}\right) \leq$ very small

3. Use Union-Bound + Subadditivity of the rank

$$\exists \Pi \text{ s.t } \operatorname{rk}(M^{\Pi}[f]) \leq \sum_{\text{small}} \operatorname{rk}(M^{\Pi}[f_i]) < \text{full-rank}$$

WHAT COMES NEXT

- Lower Bounds for general n.c circuits via the Hankel Matrix ?
- What about non-homogeneous polynomials?
- Lower Bounds for free algebras + polynomial identities $\underbrace{[x_1,x_2]\cdot[x_3,x_4]=0 \;+\; [[x_1,x_2]^2,x_1]=0}_{=M^{2\times 2}(\mathbb{F})}$
- PIT in the non-commutative setting?